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The hydrodynamical alpha-effect in a compressible medium

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The problem of the interaction of large-scale vortices with small-scale homogeneous isotropic helical turbulence in a compressible medium is considered. Averaged equations are derived using a closure procedure which is based on the functional technique. It is shown that the averaged vorticity equation has solutions that grow exponentially in time and which describe the effect of amplification of large-scale helical vortices by turbulence (hydrodynamical α-effect). The dependence of the growth rate on the compressibility is analysed, the limiting cases of incompressible fluid and turbulence δ-correlated in time being considered. The applications of the hydrodynamical α-effect discussed include the Earth's atmosphere and interstellar gas of spiral galaxies.

1. Introduction

The effect of the generation of a large-scale magnetic field by helical turbulence in a conducting medium, developed by Steenbeck, Krause & Rädler (1966), is usually called the alpha-effect (see also Moffatt 1978; Parker 1979; Krause & Rädler 1980). By analogy, we call the generation of large-scale vortices in the helical turbulence the hydrodynamical alpha-effect ($H\alpha$-effect). This effect has been revealed by Moiseev *et al.* (1983*b*) (see also Sagdeev *et al.* 1984; Tur, Khomenko & Yanovsky 1984). They have obtained the mean-field hydrodynamical equations for homogeneous isotropic helical turbulence in a compressible fluid. As shown by these authors, the linearized equation has the same form as the appropriate α-effect equation in mean-field electrodynamics:

$$\partial_t \boldsymbol{\Omega} + \alpha \nabla \times \boldsymbol{\Omega} = \nu \nabla^2 \boldsymbol{\Omega}, \tag{1}$$

where $\boldsymbol{\Omega} = \nabla \times \boldsymbol{V}$ is a mean vorticity and the uniform coefficients α and ν are related to the random velocity field parameters. The effect of the generation of large-scale vortices is associated with the term $\alpha \nabla \times \boldsymbol{\Omega}$ in (1) which leads to exponential growth of vorticity. The factor α in this term expresses the helicity of turbulent motion. The idea that the helicity of the turbulence may influence the energy transfer from small scales to large ones has been discussed by Kraichnan (1973), Brissaud *et al.* (1973), André & Lesier (1977) and Moffatt (1981) but the averaged equations were not derived in these papers and therefore the large-scale instability was not specified and discussed.

Later Sagdeev *et al.* (1987), Moiseev *et al.* (1987, 1988), Tur *et al.* (1987), Gvaramadze, Khomenko & Tur (1988), Frisch, She & Sulem (1987), and Sulem *et al.* (1989) found several examples of the α-effect in hydrodynamics, for an incompressible fluid. In all examples some additional factors, such as inhomogeneous regular flow, stable or unstable stratification, gravity force or anisotropy must supplement helical turbulence, to provide the instability. In these examples the $H\alpha$-

effect takes a much more complicated form than in a compressible fluid and is characterized by a tensorial coefficient α_{ij} instead of the scalar α in (1). It appears from the above-mentioned papers that the α-effect in hydrodynamics is as natural for helical turbulence as the effect of magnetic field generation in magnetohydrodynamics. The $H\alpha$-effect ensures energy transfer in helical turbulence from small-scale vortices to large-scale vortex structures typical of many hydrodynamic systems. Since in the compressible case considered here the coefficient α is a scalar and the $H\alpha$-equation has the simplest form, it is justified to return to this case and discuss general properties of the $H\alpha$-effect using this simple but quite general example. Moiseev *et al.* (1983*b*) considered the $H\alpha$-effect under several simplifying assumptions. The major simplification consists in the assumption that the turbulent vortical velocity field is a random process δ-correlated in time. As a result, the coefficient α proves to be independent of the compressibility of the medium. Krause & Rüdiger (1974) have shown under similar assumptions that in an incompressible fluid the $H\alpha$-effect is precluded by the symmetry of the Reynolds stresses. One is thus led to examining the status of (1) in the compressible case and the conditions when this equation is applicable. Here we present both a qualitative discussion and a quantitative derivation of the $H\alpha$-equation in compressible fluid without restriction to a δ-correlated approximation for turbulence and analyse the dependence of the generation parameter α on the compressibility of the medium. As shown below, the compressibility enters through the parameter $\mu = \lambda_{\text{cor}}/c\tau_{\text{cor}}$, where c is the sound speed, λ_{cor} and τ_{cor} are the energy-range spatial and temporal scales of the turbulence, respectively. Limits of both large and small μ are considered below. In the weak-compressibility limit ($\mu \ll 1$) we obtain $\alpha \sim \mu^2$, and α vanishes in the incompressible limit, when $c \to \infty$ (which is quite natural since there are no symmetry-breaking factors, apart from compressibility, in our problem). The opposite limit $\mu \gg 1$ is realized when the correlation time is small with $\mu \to \infty$ in the δ-correlated approximation. In this limit the factor α is independent of μ, i.e. independent of the compressibility of the medium in accordance with the result of Moiseev *et al.* (1983*b*).

The paper is organized as follows. In §2 the problem is discussed qualitatively. In §3 the derivation of the mean-field equation is outlined. In §4 the coefficients α and ν of the mean-field equation (1) are evaluated for the particular form of the correlation function of the turbulence. Their dependence on the compressibility of the medium is analysed, the incompressible limit and limit of short correlated time discussed. In §5 the large-scale instability is revealed in the derived equations. In §6 the results and applications of the $H\alpha$-effect are discussed. The details of derivation of the mean-field equation are presented in the Appendix.

2. Basic ideas and qualitative analysis

One starts from equations of motion that in a compressible medium have the form

$$\partial_t V_i + V_k \partial_k V_i = \nu_0 \nabla^2 V_i - \frac{c^2}{\rho_0} \partial_i \rho, \tag{2}$$

$$\partial_t \rho + \partial_k (V_k \rho) = 0, \tag{3}$$

where ν_0 is the kinematic viscosity, c is the sound speed and ρ is the density. For simplicity, we consider a polytropic gas with $P = \rho^\gamma$ and $\gamma = 2$. Let us assume now that in the medium described by (2) and (3) the turbulence is driven by an external stirring force (or in any other way). Let the velocity of turbulent pulsations be V';

then the velocity and the density can be separated into regular and random components:

$$V = V^{(1)}(r,t) + V'(r,t), \quad \langle V\rangle = V^{(1)}, \quad \langle V'\rangle = 0;$$
$$\rho = \rho_0 + \rho^{(1)}(r,t) + \rho'(r,t),$$
$$\langle\rho\rangle = \rho_0 + \rho^{(1)}(r,t), \quad \langle\rho'\rangle = 0;$$

where $\rho_0 = \text{const}$ is a uniform background density, $\rho^{(1)}$ and ρ' are regular and random components of its variable part. After ensemble averaging, (2) and (3) take the form

$$\partial_t V_i^{(1)} + \langle V'_k \partial_k V'_i\rangle + V_k^{(1)} \partial_k V_i^{(1)} = \nu_0 \nabla^2 V_i^{(1)} - \frac{c^2}{\rho_0} \partial_i \rho^{(1)}, \tag{4}$$

$$\partial_t \rho^{(1)} + \partial_k (V_k^{(1)} (\rho_0 + \rho^{(1)}) + \langle V'_k \rho'\rangle) = 0. \tag{5}$$

These equations contain the unknown quantities $\langle V'_k \partial_k V'_i\rangle$ and $\langle V'_k \rho'\rangle$ which we refer as 'Reynolds terms' (related to Reynolds stresses) and a closure is required. So as to express the Reynolds terms in the mean fields $V^{(1)}$, $\rho^{(1)}$ and the quantities that specify the random velocity and density fields. In the next section we do this explicitly using the functional technique while here we discuss the closure procedure qualitatively.

First of all one needs to formulate the problem:

(i) The mean values indicated by angular brackets are defined as ensemble averages. Interpretation of these averages can rely on the ergodic hypothesis.

(ii) Our main concern is evolution of the large-scale solenoidal part of the $V^{(1)}$ in the turbulent medium; hence the spatial, L, and temporal, T, scales of the field $V^{(1)}$ are assumed to be large compared to the respective energy-range scales λ and τ of the turbulence ($L \gg \lambda$; $T \gg \tau$). More generally, $V^{(1)}$ consists of many harmonics of different scales; moreover, the turbulence has the effect that an initially narrow large-scale wave packet ultimately diffuses in scales. But we ignore this in our approach, which we call the two-scale approximation.

(iii) The background turbulence (in the absence of the mean flow) is considered prescribed and we specify it as stationary, homogeneous and isotropic. All moments of such random field are invariant for a translation and rotation of the frame, i.e. there are no preferential directions or positions in the medium or is a certain time singled out.

(iv) When a large-scale perturbation is superposed, the turbulence due to nonlinear interaction with the large-scale mean flow acquires an inhomogeneous part which is described by perturbations of the Reynolds terms. The background turbulence, however, is treated as independent of the mean flow (this approach seems to be justified if the amplitude of $V^{(1)}$ is small) and it remains homogeneous and isotropic and does not contribute to the Reynolds terms. Hence, only an inhomogeneous part of the turbulence contributes to the perturbation of the Reynolds terms around their zero value.

With this formulation of the problem, it is clear that the Reynolds terms $\langle V'_k \partial_k V'_i\rangle$ are the functional of $V^{(1)}$ and generally this dependence may be complicated and nonlinear. But if the amplitude of $V^{(1)}$ is small the functional can be linearized and one can consider the Reynolds terms as a linear functional of $V^{(1)}$. The single-point average, hence, can be presented as an expansion in gradients of the large-scale field $V^{(1)}$:

$$\langle V'_k(r,t)\, \partial_k V'_i(r,t)\rangle = T_i^{(0)} + T_{ik}^{(1)} V_k^{(1)}(r,t) + T_{ikl}^{(2)} \partial_k V_l^{(1)}(r,t) + T_{iklm}^{(3)} \partial_k \partial_l V_m^{(1)}(r,t) + T_{iklmn}^{(4)} \partial_k \partial_l \partial_m V_n^{(1)}(r,t) + \dots, \tag{6}$$

where the tensorial coefficients $\boldsymbol{T}^{(i)}$ should be expressed in terms of the moments of the turbulent fields. When the medium is homogeneous, $\boldsymbol{T}^{(i)}$ are uniform and independent of position. Since the medium is isotropic, tensors $\boldsymbol{T}^{(i)}$ are invariant under rotation. They can consequently be constructed only from invariant tensors δ_{ik} and ϵ_{ijk} (where δ_{ik} is a Kronecker delta and ϵ_{ijk} is an absolutely antisymmetric tensor) and scalar parameters that specify the turbulent field (cf. Batchelor 1953). Then, the solenoidal part of the Reynolds terms can be written in the form ($T_i^{(0)}$ vanishes because Reynolds terms have to vanish when $\boldsymbol{V}^{(1)}$ is identically zero):

$$\begin{aligned}\langle V'_k(\boldsymbol{r},t)\,\partial_k V'_i(\boldsymbol{r},t)\rangle &= C_0 V_i^{(1)} + C_1 \epsilon_{ijk}\partial_j V_k^{(1)} + C_2 \nabla^2 V_i^{(1)} + C_3 \nabla^2 \epsilon_{ijk}\partial_j V_k^{(1)} \\ &\quad + C_4 \nabla^4 V_i^{(1)} + \dots \\ &= \epsilon_{ijk}\partial_j (C_1 + C_3\nabla^2 + C_5 \nabla^4 + \dots + C_{2n+1}\nabla^{2n} + \dots) V_k^{(1)} \\ &\quad + (C_0 + C_2 \nabla^2 + C_4\nabla^4 + \dots + C_{2n}\nabla^{2n} + \dots) V_i^{(1)}, \end{aligned} \tag{7}$$

where the coefficients $C_0, C_2, C_4, \dots, C_{2n}$ with even subscripts are scalar constants and the coefficients $C_1, C_3, C_5, \dots, C_{2n+1}$ with odd subscripts are pseudoscalars. It is clear that the average $\langle V'_k \partial_k V'_i\rangle$ is a polar vector, and the quantity $C_1 \epsilon_{ijk}\partial_j V_k^{(1)}$ is also a polar vector only when C_1 is a pseudoscalar. In the next section we derive the equation which governs the evolution of large-scale vortices under influence of small-scale helical turbulence. It is clear from (6) and (7) that the helicity alone (or another non-invariant for the space reversal characteristic) is insufficient for the $H\alpha$-effect but additional violation of symmetry (e.g. compressibility) is required. In fact, in the incompressible case (when $\partial_k V_k = 0$) the nonlinear term $\langle V'_k \partial_k V'_i\rangle$ can be represented as $\partial_k\langle V'_k V'_i\rangle$. The tensor $\langle V'_k V'_i\rangle$ is symmetric in subscripts and one immediately concludes that the tensor $T_{ikl}^{(2)}$ is also symmetric in subscripts i, k and cannot be proportional to ϵ_{ikl}; hence, all coefficients C_{2n+1} must vanish. A tensor of rank three cannot be constructed from δ_{ik} alone, hence the $H\alpha$-effect is precluded in isotropic homogeneous turbulence of an incompressible fluid (Krause & Rüdiger 1974).

Nevertheless, the $H\alpha$-effect in helical turbulence in an incompressible fluid does exist, provided some symmetry of the problem is violated. The role of the symmetry-breaking factor can be played by, for example, the inhomogeneous steady flow (Tur *et al.* 1987; Gvaramadze *et al.* 1989) or a temperature gradient in a gravity field (Sagdeev *et al.* 1987; Moiseev *et al.* 1988). In these cases tensors $\boldsymbol{T}^{(i)}$ are free from the restriction that they must be symmetric under rotation and translation of the frame. They can involve not only ϵ_{ikl} and δ_{ik} but also the quantities that specify the imposed inhomogeneity and anisotropy. A similar situation occurs in the anisotropic kinetic alpha-effect of Frisch *et al.* (1987) who considered the background of a parity-non-invariant turbulent velocity field.

In the compressible case the nonlinear term cannot be represented as $\partial_k\langle V'_k V'_i\rangle$, therefore the tensor $T_{ikl}^{(2)}$ need not be symmetric in subscripts and the non-zero α-term is not precluded. In the next section we derive the averaged equation and obtain an explicit expression for the coefficient α, while here we estimate α using dimensional considerations.

According to (ii) in the formulation of the problem given above, we split the random component of the velocity $\boldsymbol{V}'$ into two parts: $\boldsymbol{V}' = \boldsymbol{V}^{\mathrm{t}} + \tilde{\boldsymbol{V}}$, where $\boldsymbol{V}^{\mathrm{t}}$ denotes the background turbulence, defined in (iii), and $\tilde{\boldsymbol{V}}$ refers to the inhomogeneous part of the turbulence which arises owing to the interaction between $\boldsymbol{V}^{(1)}$ and $\boldsymbol{V}^{\mathrm{t}}$. The equation which governs the perturbation $\tilde{\boldsymbol{V}}$ will be written in the next section, while the properties of $\boldsymbol{V}^{\mathrm{t}}$ are prescribed by (iii). Note that the background turbulence alone does not contribute to the Reynolds term. Indeed, the single-point correlation

$\langle V_k^t \partial_k V_i^t \rangle$ vanishes when V^t is a homogeneous random vector field. Then $\langle V_k' \partial_k V_i' \rangle$ in (6) and (7) is now understood as a sum $\langle \tilde{V}_k \partial_k V_i^t + V_k^t \partial_k \tilde{V}_i + \tilde{V}_k \partial_k \tilde{V}_i \rangle$. The constants C_i can be represented as infinite series in unperturbed correlation functions of increasing orders. In turn, each term in this series is an averaged quantity featuring the unperturbed velocity field contracted with invariant tensors δ_{ik} and ϵ_{ijk}. The form of such expressions is rather bulky so we evaluate the coefficients C_i in a second-order correlation approximation. In this approach we assume that the second-order correlation function provides a sufficiently good description of the turbulence.

The correlation tensor of the unperturbed homogeneous isotropic turbulence can be written in the form (see e.g. Monin & Yaglom 1975)

$$\langle V_i^t(\boldsymbol{r}_1, t_1)\, V_j^t(\boldsymbol{r}_2, t_2)\rangle, = \mathrm{A}(\mathrm{r}, \tau)\,\delta_{ij} + B(r,\tau)\, r_i r_j + H(r,\tau)\,\epsilon_{ijk} r_k, \quad \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2, \quad \tau = t_1 - t_2. \tag{8}$$

The correlation function (8) contains a single pseudoscalar function $H(r,\tau)$, which represents the helicity.

$$H(0,0) = \tfrac{1}{6}\langle \boldsymbol{V}^t(\boldsymbol{r},t)\cdot \boldsymbol{\nabla} \times \boldsymbol{V}^t(\boldsymbol{r},t)\rangle.$$

In our formulation of the problem (see (iv) above) a non-vanishing contribution to the Reynolds terms arises only when the turbulence becomes inhomogeneous, which is possible only if the mean velocity is assumed non-uniform. Therefore if $\boldsymbol{V}^{(1)}$ were to be uniform the Reynolds terms would vanish; hence, C_0 vanishes. From dimensional considerations,

$$C_1 \sim \int H(0,\tau)\,\mathrm{d}\tau \sim H(0,\ 0)\,\tau_{\mathrm{cor}}, \tag{9}$$

$$C_2 \sim \int A(0,\tau)\,\mathrm{d}\tau \sim A(0,0)\,\tau_{\mathrm{cor}}, \tag{10}$$

with the understanding that the coefficients of the expansion (7) rapidly decrease. Indeed, from dimensional arguments we see that $C_3 \sim \lambda^2 C_1$; $\nabla^2 \boldsymbol{V}^{(1)} \sim L^{-2}\boldsymbol{V}^{(1)}$, where λ is characteristic spatial scale of functions A, B, and H in (8), and L is a spatial scale of mean velocity $\boldsymbol{V}^{(1)}$. Therefore,

$$C_3 \nabla^2 \partial_j V_k^{(1)} \sim (\lambda/L)^2 C_1 \partial_j V_k^{(1)},$$
$$C_4 \nabla^4 V_i^{(1)} \sim (\lambda/L)^2 C_2 \nabla^2 V_i^{(1)},$$
$$\vdots$$

That is (6) is actually an expansion in a small parameter λ/L and the higher terms can be neglected.

It appears from (7) that the term which is proportional to C_1 is responsible for the $H\alpha$-effect. The next term, proportional to C_2, contributes to the dissipation in the vorticity equation and (10) presents an estimate of the turbulent viscosity. Comparing (1) and (7) leads to $\alpha \sim C_1$ and $\nu \sim C_2$. It is apparent from (7) that in order to give rise to the $H\alpha$-effect, the turbulence has to be parity-non-invariant. Let us discuss this crucial point in a little more detail.

When we say that the turbulence is lacking parity-invariance we always imply that some statistical parameters of such turbulence are non-invariant under reflection of spatial variables, while all moments of the turbulent field, of course, remain true tensors. One can distinguish, consequently, the right-handed frame from the left-handed one by measuring these parameters, e.g. the correlation function of the scalar product of the velocity and vorticity $\langle \boldsymbol{V}\cdot\boldsymbol{\nabla}\times\boldsymbol{V}\rangle$. A familiar example of turbulence lacking parity-invariance is provided by helical turbulence.

In addition we would like to note that the discussion presented in this section should not be regarded as a rigorous proof, but as a rule of thumb, which does not contain all the peculiarities inherent in the compressible case. In the next section we take into account compressibility and sketch the derivation of the averaged equation which governs the $H\alpha$-effect in a compressible medium. A more strict and detailed derivation, based on the variational technique, is presented in the Appendix.

3. Closure of the averaged equation and the emergence of the α-term

Consider the turbulence in a compressible medium driven by a random stirring force $\boldsymbol{F}'$ ($\langle \boldsymbol{F}' \rangle = 0$). The random velocity $\boldsymbol{V}^{\mathrm{t}}$ ($\langle \boldsymbol{V}^{\mathrm{t}} \rangle = 0$) obeys

$$\partial_t V_i^{\mathrm{t}} + V_k^{\mathrm{t}} \partial_k V_i^{\mathrm{t}} = \nu_0 \nabla^2 V_i^{\mathrm{t}} - \frac{c^2}{\rho_0} \partial_i \rho^t + F_i', \tag{11}$$

$$\partial_t \rho^{\mathrm{t}} + \partial_k (V_k^{\mathrm{t}} \rho^{\mathrm{t}}) = 0, \tag{12}$$

where $\rho_0 = \mathrm{const}$ is the uniform background density, ρ^{t} is the random pulsations of density associated with turbulent motion ($\langle \rho \rangle = \rho_0, \langle \rho^{\mathrm{t}} \rangle = 0$). The external random force $\boldsymbol{F}'$ is assumed to be homogeneous and isotropic so that the correlation tensor (or a covariance matrix) of the field $\boldsymbol{V}^{\mathrm{t}}$ has the form (8), while the correlator $\langle \rho^{\mathrm{t}} \boldsymbol{V}^{\mathrm{t}} \rangle$ vanishes. Thus, the homogeneous isotropic stationary helical turbulent fields $\boldsymbol{V}^{\mathrm{t}}$ and ρ^{t} governed by (11) and (12) are regarded as an unperturbed state.

Let us introduce a large-scale vortex perturbation $\boldsymbol{V}^{(1)}$ into such turbulence. This perturbation interacts with the turbulence, producing thereby inhomogeneous random components $\tilde{\boldsymbol{V}}$ and $\tilde{\rho}$. Then the velocity and the density fields can be represented as

$$\left.\begin{aligned} &\boldsymbol{V} = \boldsymbol{V}^{(1)}(\boldsymbol{r},t) + \boldsymbol{V}^{\mathrm{t}}(\boldsymbol{r},t) + \tilde{\boldsymbol{V}}(\boldsymbol{r},t), \\ &\langle \boldsymbol{V} \rangle = \boldsymbol{V}^{(1)}(\boldsymbol{r},t), \quad \langle \boldsymbol{V}^{\mathrm{t}} \rangle = \langle \tilde{\boldsymbol{V}} \rangle = 0; \\ &\rho = \rho_0 + \rho^{(1)}(\boldsymbol{r},t) + \rho^{\mathrm{t}}(\boldsymbol{r},t) + \tilde{\rho}(\boldsymbol{r},t), \\ &\langle \rho \rangle = \rho_0 + \rho^{(1)}(\boldsymbol{r},t), \quad \langle \rho^{\mathrm{t}} \rangle = \langle \tilde{\rho} \rangle = 0. \end{aligned}\right\} \tag{13}$$

Mean fields $\boldsymbol{V}^{(1)}(\boldsymbol{r},t)$ and $\rho^{(1)}(\boldsymbol{r},t)$ are described by the averaged equations

$$\partial_t V_i^{(1)} + \langle \tilde{V}_k \partial_k V_i^{\mathrm{t}} \rangle + \langle V_k^{\mathrm{t}} \partial_k \tilde{V}_i \rangle = \nu_0 \nabla^2 V_i^{(1)} - \frac{c^2}{\rho_0} \partial_i \rho^{(1)}, \tag{14}$$

$$\partial_t \rho^{(1)} + \partial_k (\langle V_k^{\mathrm{t}} \tilde{\rho} \rangle + \langle \tilde{V}_k \rho^{\mathrm{t}} \rangle) = 0. \tag{15}$$

Thus, we assume that V^{t} and ρ^{t} continue to obey equations (11) and (12) while all inhomogeneous variations of the random fields are described by functions $\tilde{\boldsymbol{V}}(\boldsymbol{r},t)$ and $\tilde{\rho}(\boldsymbol{r},t)$ through equations

$$\partial_t \tilde{V}_i - \nu_0 \nabla^2 \tilde{V}_i + \frac{c^2}{\rho_0} \partial_i \tilde{\rho} = -V_k^{(1)} \partial_k V_i^{\mathrm{t}} - V_k^{\mathrm{t}} \partial_k V_i^{(1)}, \tag{16}$$

$$\partial_t \tilde{\rho} + \rho_0 \partial_k \tilde{V}_k = -\partial_k (V_k^{(1)} \rho^{\mathrm{t}} + V_k^{\mathrm{t}} \rho^{(1)}). \tag{17}$$

In these equations the terms nonlinear in perturbations $\boldsymbol{V}^{(1)}$, $\rho^{(1)}$ and $\tilde{\boldsymbol{V}}$, $\tilde{\rho}$ are supposed to be small and are omitted (this seems to be justified at least when the amplitudes of the mean fields $\boldsymbol{V}^{(1)}$ and $\rho^{(1)}$ are small).

Averaged equations (14) and (15) involve unknown terms contained in the angular brackets (the Reynolds terms) and a closure is required. Here we use the simplified approach to concentrate on the physical picture of the effect, leaving a more strict consideration to the Appendix, where the closure based on the Furutsu–Novikov formula is performed. As we see above, the general form of the linearized averaged equation is

$$\partial_t V + \alpha \nabla \times V = \nu \nabla^2 V. \tag{18}$$

Our purpose now is to demonstrate the emergence of the term $\alpha\nabla\times V$ from the basic equations. Indeed, the averaged equation (14) involves the Reynolds terms $\langle \tilde{V}_k \partial_k V_i^{\mathrm{t}} \rangle + \langle V_k^{\mathrm{t}} \partial_k \tilde{V}_i \rangle$ which can be written in the form

$$\langle \tilde{V}_k \partial_k V_i^{\mathrm{t}} \rangle + \langle V_k^{\mathrm{t}} \partial_k \tilde{V}_i \rangle = \partial_k(\langle \tilde{V}_k V_i^{\mathrm{t}} \rangle + \langle V_k^{\mathrm{t}} \tilde{V}_i \rangle) - \langle V_i^{\mathrm{t}} \nabla\cdot \tilde{V} \rangle - \langle \tilde{V}_i \nabla\cdot V^{\mathrm{t}} \rangle. \tag{19}$$

The first term on the right-hand side of (19) cannot lead to a term of the type $\alpha\nabla\times V$ in the averaged equation because the tensor $(\langle \tilde{V}_k V_i^{\mathrm{t}} \rangle + \langle V_k^{\mathrm{t}} \tilde{V}_i \rangle)$ is symmetric in subscripts i and k. The mean value $\langle \tilde{V}_i \nabla\cdot V^{\mathrm{t}} \rangle$ cannot result in an α-term because the helical part of the correlation function of the background turbulence V^{t}, represented by the last term in (8)

$$\langle V_i^{\mathrm{t}} V_j^{\mathrm{t}} \rangle^{(\mathrm{h.p.})} = H(\boldsymbol{r}, \tau)\, \epsilon_{ijk} r_k,$$

fits the condition $\partial_j \langle V_i^{\mathrm{t}} V_j^{\mathrm{t}} \rangle^{(\mathrm{h.p.})} = \langle V_i^{\mathrm{t}} \nabla\cdot V^{\mathrm{t}} \rangle = 0$ by construction. Hence, it is the mean value $\langle V_i^{\mathrm{t}} \nabla\cdot \tilde{V} \rangle$ which gives rise to the α-term in the averaged equation. A scalar quantity $\varphi = \nabla\cdot\tilde{V}$ obeys the wave equation with a stirring force:

$$\partial_t^2 \varphi - c^2 \nabla^2 \varphi - \nu_0 \nabla^2 \partial_t \varphi = -2\partial_t(\partial_m V_i^{(1)})(\partial_i V_m^{\mathrm{t}}). \tag{20}$$

The right-hand side of (20) varies at the high frequency $\omega \sim 1/\tau_{\mathrm{cor}}$ and its amplitude is modulated by a slow function $V^{(1)}$. Therefore, φ oscillates at a higher frequency of the same order as that of V^{t}. The approximate solution of this equation can be easily derived by using the Fourier transformation in fast variables:

$$\varphi(\boldsymbol{k}, \omega) \approx -2\partial_m V_i^{(1)} \frac{\omega k_i V_m^{\mathrm{t}}(\boldsymbol{k}, \omega)}{c^2 k^2 - \omega^2 + \mathrm{i}\nu_0 k^2 \omega}.$$

Multiplying this with V_j^{t} and averaging over an ensemble one finds

$$\langle V_j^{\mathrm{t}}(\boldsymbol{k}_1, \omega_1) \varphi(\boldsymbol{k}, \omega) \rangle = -2\partial_m V_i^{(1)} \frac{\omega k_i \langle V_j^{\mathrm{t}}(\boldsymbol{k}_1, \omega_1) V_m^{\mathrm{t}}(\boldsymbol{k}, \omega) \rangle}{c^2 k^2 - \omega^2 + \mathrm{i}\nu_0 k^2 \omega}.$$

After inverse Fourier transformation we obtain the one-point average

$$\langle V_j^{\mathrm{t}}(\boldsymbol{r}, t) \varphi(\boldsymbol{r}, t) \rangle = -2\partial_m V_i^{(1)}(\boldsymbol{r}, t) \times \int \exp\{\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}_1)\boldsymbol{r} - \mathrm{i}(\omega+\omega_1)t\} \frac{\omega k_i \langle V_j^{\mathrm{t}}(\boldsymbol{k}_1, \omega_1) V_m^{\mathrm{t}}(\boldsymbol{k}, \omega) \rangle}{c^2 k^2 - \omega^2 + \mathrm{i}\nu_0 k^2 \omega} \mathrm{d}\boldsymbol{k}_1 \,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega_1\,\mathrm{d}\omega.$$

The final result is reached on substituting the correlation function of form (A 6) and after slight adjustment of notation $(\nabla\cdot\tilde{V} = \varphi)$ reads

$$\langle V_j^{\mathrm{t}}(\boldsymbol{r}, t) \nabla\cdot \tilde{V}(\boldsymbol{r}, t) \rangle = -2\epsilon_{jmn} \partial_m V_i^{(1)}(\boldsymbol{r}, t) \int \frac{\mathrm{i}\omega k_i k_n G(k, \omega)}{c^2 k^2 - \omega^2 + \mathrm{i}\nu_0 k^2 \omega} \mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega.$$

Performing the integration over the angles this reduces to

$$\langle V_j^{\mathrm{t}}(\boldsymbol{r}, t) \nabla\cdot \tilde{V}(\boldsymbol{r}, t) \rangle = \alpha \nabla \times V^{(1)}(\boldsymbol{r}, t),$$

where α is defined by

$$\alpha = -\tfrac{8}{3}\pi \int_{-\infty}^{+\infty} \mathrm{d}\omega \int_0^{\infty} k^4\, \mathrm{d}k \frac{G(k, \omega)\, \mathrm{i}\omega}{c^2 k^2 - \omega^2 + \mathrm{i}\omega\nu_0 k^2}, \tag{21}$$

where $G(k,\omega)$ is the helicity-related coefficient in the Fourier transformation of the correlation function (see (A 5) and (A 6)). Thus, the α-term in the mean-field equation is associated with the non-uniform scalar field $\varphi = \nabla\cdot\tilde{V}$ which contributes to the solenoidal part of the average $\langle V^t\nabla\cdot\tilde{V}\rangle$. Since the quantity $\alpha\nabla\times V^{(1)}$ is a slow one, higher-frequency components of the field φ are compensated by higher-frequency components of the field V^t which always exist in a continuous spectrum of the turbulence.

The turbulent viscosity has been evaluated by Krause & Rüdiger (1974) and is given by

$$\nu = \nu_0 + \tfrac{16}{15}\pi\int_{-\infty}^{+\infty}\mathrm{d}\omega\int_0^{\infty}\mathrm{d}k k^2\frac{2\nu_0 k^2-\mathrm{i}\omega}{(\nu_0 k^2-\mathrm{i}\omega)^2}D(k,\omega), \tag{22}$$

where $D(k,\omega)$ is the scalar coefficient in the Fourier transformation of the correlation function (see (A 5) and (A 6)).

Equation (18) is the desired result of the closure of the averaged equation, while (21) and (22) for α and ν provide explicit expressions for coefficients C_1 and C_2 in the expansion (7).

4. Dependence of α on compressibility and some limiting cases

Let us consider the structure of (21) in more detail. The integrand depends parametrically on the sound speed c and the kinematic viscosity ν_0, while helicity $G(k,\omega)$ depends parametrically on characteristic scales of the turbulence τ_{cor} and λ_{cor} ($G(k,\omega) = G(k\lambda_{\mathrm{cor}}, \omega\tau_{\mathrm{cor}})$, as follows from dimensional arguments).

Further suppose that the spectral density of the correlation function of velocity and vorticity $G(k,\omega)$ factorizes:

$$G(k,\omega) = g_0\,G_1(\omega)\,G_2(k), \tag{23}$$

where g_0 is a dimensional constant and G_1 and G_2 are dimensionless functions. (Since we consider a prescribed turbulence which is exited by a stirring force, this assumption amounts to a special choice of the random force.) As an example, consider the function $G_1(\omega)$ of the form

$$G_1(\omega) = (\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{\omega_{\mathrm{cor}}^2}{\omega^2+\omega_{\mathrm{cor}}^2}, \tag{24}$$

where $\omega_{\mathrm{cor}} \sim 1/\tau_{\mathrm{cor}}$. In the limit $\tau_{\mathrm{cor}}\rightarrow 0$ ($\omega_{\mathrm{cor}}\rightarrow\infty$) the spectral density tends to a constant:

$$\lim_{\omega_{\mathrm{cor}}\rightarrow\infty} G_1(\omega) = (\tfrac{1}{2}\pi)^{\frac{1}{2}} = \text{const.}$$

This limiting case corresponds to the white noise or a random process δ-correlated in time. Indeed, the inverse Fourier transformation of (24) reads

$$\begin{aligned} G_1(\tau) &= (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{+\infty}\mathrm{d}\omega\exp(-\mathrm{i}\omega\tau)\,G_1(\omega) = \tfrac{1}{2}\omega_{\mathrm{cor}}\exp(-\omega_{\mathrm{cor}}|\tau|) \\ &= \tfrac{1}{2}\tau_{\mathrm{cor}}^{-1}\exp(-|\tau|/\tau_{\mathrm{cor}}), \quad \lim_{\tau_{\mathrm{cor}}\rightarrow 0} G_1(\tau) = \delta(\tau). \end{aligned}$$

Substituting (23) and (24) into (21) and performing the integration over ω one arrives at the following expression for α:

$$\alpha = \pi g_0\int_0^{\infty}k^4\,\mathrm{d}k G_2(k)\frac{\omega_{\mathrm{cor}}^2}{(\nu_0\,\omega_{\mathrm{cor}}+c^2)\,k^2+\omega_{\mathrm{cor}}^2}. \tag{25}$$

Notice that in the limit $\omega_{cor} \to \infty$ (i.e. $\tau_{cor} \to 0$) this reduces to

$$\alpha \sim \pi g_0 \int_0^\infty k^4 \, dk G_2(k) \sim H(0,0) \sim \langle V^t(r,t) \cdot \nabla \times V^t(r,t) \rangle$$

in agreement with the estimate (9). To perform the integration over k in (25), the form of the function $G_2(k)$ should be specified, for example as

$$G_2(k) = \exp(-\lambda_{cor}^2 k^2). \tag{26}$$

Substitution of this into (25) yields

$$\alpha = \tfrac{1}{4}\pi^{\frac{3}{2}} g_0 \lambda_{cor}^{-5} \mu^2 \{1 - 2\mu^2 + 2\pi^{\frac{1}{2}} \mu^3 \exp(\mu^2) [1 - \phi(\mu)]\}, \tag{27}$$

where

$$\phi(\mu) = 2\pi^{-\frac{1}{2}} \int_0^\mu \exp(-t^2) \, dt.$$

Here μ is the dimensionless parameter given by

$$\mu^2 = \frac{\omega_{cor}^2 \lambda_{cor}^2}{\nu_0 \omega_{cor} + c^2} = \frac{\lambda_{cor}^2}{\nu_0 \tau_{cor} + c^2 \tau_{cor}^2}. \tag{28}$$

Equation (27) for α is exact for the adopted model correlation function specified by (23), (24) and (26). Let us analyse (27) in two limiting cases; for $\mu \gg 1$ and $\mu \ll 1$. In the former case using the asymptotic form of the error function $\phi(\mu)$ (see e.g. Janke, Emde & Lösch 1960) we obtain

$$\alpha \approx \tfrac{3}{8}\pi^{\frac{3}{2}} g_0 \lambda_{cor}^{-5} (1 - \tfrac{5}{2}\mu^{-2}) \quad \text{for} \quad \mu \gg 1. \tag{29}$$

In the opposite limiting case, $\mu \ll 1$, (27) reduces to

$$\alpha \approx \tfrac{1}{4}\pi^{\frac{3}{2}} g_0 \lambda_{cor}^{-5} (\mu^2 - 2\mu^4) \quad \text{for} \quad \mu \ll 1.$$

It can be seen from (29) that the leading term in the asymptotic expansion of α for large μ is independent of μ. This limiting case corresponds to a random process δ-correlated in time. Indeed, it is apparent from (28) that $\mu \to \infty$ when $\tau_{cor} \to 0$ provided that the values of ν_0 and c are finite. In this case one is led to $\alpha \sim H(0,0)$, which recovers the result of Moiseev *et al.* (1983*b*).

When $\nu_0 \ll c^2 \tau_{cor}$ the viscosity in (28) can be neglected and it becomes clear that $\mu \sim (\lambda_{cor}/\tau_{cor})/c$ is an analogue of the Mach number. The limit $\tau_{cor} \to 0$ with $\mu \to \infty$ is formally equivalent to the limit $c \to 0$, $\mu \to \infty$. Therefore, when the turbulence is considered as a random process δ-correlated in time, provided that c is finite, α becomes independent of the sound speed.

The transition to an incompressible fluid can be accomplished by putting $c \to \infty$ provided that τ_{cor} is finite. Then $\mu \to 0$ when $c \to \infty$ and α vanishes in this limit. One is led consequently to the already known conclusion that in incompressible isotropic homogeneous turbulence the $H\alpha$-effect cannot be realized.

5. The large-scale instability

The linearized averaged equation (18) has unstable solutions which describe the generation of the large-scale vortices. This can be easily shown using the Fourier representation. In fact, looking for a solution $V^{(1)}(k) = (V_x^{(1)}(k);\ V_y^{(1)}(k);\ 0)$ with the wave vector $\boldsymbol{k}$ parallel to the z-axis, one obtains for the growth rate

$$\gamma = \alpha k - \nu k^2.$$

The mode with the wave vector $k = k_0 = \alpha/2\nu$ grows most rapidly. Its growth rate is $\gamma_{\max} = \alpha^2/4\nu$, where ν is the turbulent viscosity which is proportional to the energy of the turbulence, while α is proportional to the helicity of the turbulent velocity field. Hence the characteristic scale $L = k_0^{-1}$ of the instability is

$$L \sim \nu/\alpha.$$

When the compressibility parameter μ is large ($\mu \gg 1$), coefficient α is independent of μ and the scale of instability is

$$L \sim \frac{\langle (V^t)^2\rangle}{\langle V^t \nabla \times V^t\rangle} \sim \frac{\int \langle (V^t)^2\rangle \mathrm{d}\boldsymbol{r}}{\int \langle V^t \nabla \times V^t\rangle \mathrm{d}\boldsymbol{r}} \sim \frac{E}{I},$$

where I is the topological invariant and E is the energy invariant. These quantities are closely associated with intrinsic properties of the turbulence and are conserved in an ideal fluid.

When compressibility is weak (i.e. $\mu \ll 1$) the scale of the instability becomes larger, acquiring the factor μ^{-2}:

$$L \sim \mu^{-2} E/I, \quad \mu \ll 1.$$

The structures generated by helical turbulence can be considered as large-scale ones provided $L \gg \lambda_{\text{cor}}$, i.e. when $E/I \gg \lambda_{\text{cor}}$. This inequality restricts the values of parameters that specify the correlation function of the turbulence. When $L \gg \lambda_{\text{cor}}$ and $\langle (V^t)^2\rangle \sim \lambda^2/\tau^2$ we find $T \sim \gamma^{-1} \gg \tau_{\text{cor}}$, that is the large-scale structure is automatically a slow one.

Note an important feature of the structures generated: they are helical themselves because they are characterized by a non-vanishing scalar product ($V^{(1)} \cdot \nabla \times V^{(1)}$).

6. Discussion and applications

As we have seen, the averaged equation (18) has unstable solutions associated with the α-term which provide the positive feedback between different components of the large-scale vector field. A question arises as to which role is played by the compressibility in the generation of a purely solenoidal field by solenoidal turbulence. To answer this question let us remember that the α-term is associated with the average $\langle V^t \nabla \cdot \tilde{V}\rangle$ and return to equation (21) for α. There are two contributions to the integral (21): the first arises from the singular points of the correlation function and the second is associated with the poles of the Green's function (i.e. the roots of the denominator). Both of them correspond to the resonances of the scalar field $\nabla \cdot \tilde{V}$ and the vector field V^t. Note that for the model correlation function specified by (23), (24) and (26) these contributions have the same order of magnitude. In the white-noise limit (or a process δ-correlated in time) we have $G(\omega) =$ constant and the singular points of the Green's function solely contribute to the integral. Furthermore, this contribution does not depend on the position of the pole and thereby is independent of the sound speed. c.

The α-term in the averaged equation ensures a positive feedback between different components of the solenoidal velocity field. The appearance of the α-term in this equation is associated with the helicity of small-scale turbulence rather than with compressibility; the compressibility plays the role of a symmetry-breaking factor that changes the symmetry of the basic equations. As mentioned in §2 there are several possible symmetry-breaking factors that can lead to the $H\alpha$-effect in an incompressible fluid. (Examples of the $H\alpha$-effect in an incompressible fluid have been considered in detail by Gvaramadze *et al.* 1989 and by Moiseev *et al.* 1988.)

Helicity is one of the most important characteristics of any vector field. The integral over the liquid volume, $I = \int (\boldsymbol{V} \cdot \nabla \times \boldsymbol{V})\, \mathrm{d}\boldsymbol{r}$, is an integral of motion in an ideal fluid and characterizes linkages of vortex lines. Note that the invariant I is a pseudoscalar, hence all fluid motions for which $I \neq 0$ lack parity-invariance.

Since helical turbulence can give birth to large-scale structures, a laboratory experiment concerning this effect would be very interesting. Unfortunately, we are not aware of any such experiments and so natural phenomena are a unique source of experimental information. As discussed in detail by Moiseev *et al.* (1983*a*), tropical cyclones (typhoons) can be an example of such helical structures in the atmosphere. Their arguments are the following. It is well-known that the air streamlines in a typhoon are linked (and the flow is helical) because intense horizontal circulation is complemented by a weaker poloidal one which is essential for the existence of typhoons (see e.g. Riehl 1976). Furthermore, it is commonly accepted that typhoons gain their energy from small-scale turbulent convection ($\lambda \sim 10$ km). Under the influence of the Coriolis force the convection, which can be regarded as a small-scale random background, becomes helical. Estimates of the growth rate and characteristic scale of the instability are in qualitative agreement with observations of typhoons. Moreover, this model explains the anticyclonic (cyclonic) character of typhoons in the northern (southern) hemisphere and gives a reasonable estimate of the geographical latitudes of the regions of active cyclogenesis. The latitude threshold appears owing to the vanishing of the mean helicity of atmospheric convection at the equator. Sagdeev *et al.* (1987) and Moiseev *et al.* (1988) advanced the theory of cyclogenesis further by taking into account the stratification of the atmosphere, heat transfer phenomena and convection.

We should note that Levich & Tzvetkov (1984, 1985) also propose a mechanism of energy transfer from small-scale motions in the atmosphere to large-scale ones which is based on the helical properties of the turbulence. However, in contrast to our approach, which relies on non-vanishing mean helicity of the turbulence, their concept employs helicity fluctuations with vanishing mean helicity.

In this paper we have evaluated the Reynolds terms in a two-scale approach, which is used at the last step in evaluating the integral in (A 19). To put it a different way, the small parameter λ/L is used to neglect high derivatives of mean velocity with respect to spatial coordinates (cf. (7) and (9), (10)). We regard this approximation as a first step in the solution of the problem of the interaction of averaged fields with random fields when their scales are different. A more realistic approach has to include the energy distribution over all scales and can be developed by using the diagram technique. Nevertheless, the form of the averaged equation for mean motions of small amplitude presumably would not change, and only the coefficients α and ν in this equation will become somewhat different. We are inclined to consider our results as an indirect manifestation of the energy transfer toward larger scales in helical turbulence.

As possible applications of the $H\alpha$-effect in a compressible medium we could mention hydrodynamic effects in galactic disks and the generation of vortices in jets in active galaxies. It is widely accepted (see e.g. Parker 1979; Ruzmaikin, Shukurov & Sokoloff 1988) that magnetic fields of spiral galaxies are associated with the α-

effect of small-scale motions. Large-scale helical vortices, hence, have to be generated as well as large-scale magnetic fields in these objects. In accordance with observations, small-scale turbulent motions in galactic disks are characterized by $\tau_{cor} \sim 3\times 10^{14}$ s, $\lambda_{cor} \sim 3\times 10^{20}$ cm, sound speed $c \sim 10^6$ cm/s and helicity $\langle V^t \cdot \nabla \times V^t \rangle \tau_{cor} \sim 10^5$ cm/s (Ruzmaikin *et al.* 1988). Then our estimate gives $\gamma \sim 0.25\times 10^{-16}$ s^{-1}, $k_0 \sim 0.5\times 10^{-21}$ cm^{-1}. That is helical vortices of the scale $L \sim 1$ kiloparsec arise in galactic disks in a time $T \sim 10^9$ years, i.e. at the same rate as large-scale magnetic fields. Since the thickness of the disk is of order 0.25 kiloparsec, the scale L has to be regarded as a rough estimate of the horizontal size of a vortex. To get a more precise estimate of these parameters one has to solve the boundary problem.

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Appendix

Here we give a more detailed derivation of the averaged equation (18). First of all define direct and inverse Fourier transforms:

$$\left.\begin{aligned} V(\boldsymbol{k},\omega) &= (2\pi)^{-2}\int \exp\,(\mathrm{i}\omega t - \mathrm{i}\boldsymbol{kr})\; V(\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}t, \\ V(\boldsymbol{r},t) &= (2\pi)^{-2}\int \exp\,(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{kr})\; V(\boldsymbol{k},\omega)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega. \end{aligned}\right\} \tag{A 1}$$

Applying the curl-operator to (14) and performing the Fourier transformation one obtains

$$(-\mathrm{i}\omega + \nu_0 k^2)\,\Omega_i(\boldsymbol{k},\omega) = R_i(\boldsymbol{k},\omega), \tag{A 2}$$

$$\begin{aligned} R_i(\boldsymbol{k},\omega) = \epsilon_{ipq} k_p \int & \delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k})\,\delta(\omega_1+\omega_2-\omega) \\ & \times (k_{1j}\delta_{qm} + k_{2m}\delta_{qj})\,\langle \tilde{V}_j(\boldsymbol{k}_2,\omega_2)\,V^t_m(\boldsymbol{k}_1,\omega_1)\rangle\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2) \end{aligned} \tag{A 3}$$

where R_i is the contribution of the Reynolds terms to the averaged vorticity equation. The two-point average $\langle \tilde{V}_j(\boldsymbol{k}_2,\omega_2) V^t_m(\boldsymbol{k}_1,\omega_1)\rangle$ can be found, using the Furutsu–Novikov formula which expresses the mean value of a product of the random process φ and its functional $F[\varphi]$, in a convolution of the correlation function of the process φ and the mean value of the functional derivative of F with respect to φ:

$$\langle F[\varphi]\varphi\rangle = \int \langle\varphi\varphi\rangle \left\langle \frac{\delta F[\varphi]}{\delta\varphi}\right\rangle \mathrm{d}t'.$$

In our case $\tilde{V}$ is a functional of the random processes V^t and ρ^t. Correlation functions of these processes are known and one needs to evaluate the averaged functional derivative $\langle \delta\tilde{V}/\delta V^t\rangle$ and substitute it in the Furutsu–Novikov formula, which in our case reads

$$\langle \tilde{V}_j(\boldsymbol{k}_2,\omega_2)\,V^t_m(\boldsymbol{k}_1,\omega_1)\rangle = \int \langle V^t_m(\boldsymbol{k}_1,\omega_1)\,V^t_n(\boldsymbol{k}_3,\omega_3)\left\langle \frac{\delta \tilde{V}_j(\boldsymbol{k}_2,\omega_2)}{\delta V^t_n(\boldsymbol{k}_3,\omega_3)}\right\rangle \mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\omega_3, \tag{A 4}$$

where $\langle V^t_j(\boldsymbol{k}_2,\omega_2) V^t_m(\boldsymbol{k}_3,\omega_3)\rangle$ is the correlator of the unperturbed turbulence in the Fourier representation. (The term $\int\langle V^t\rho^t\rangle\langle\delta\tilde{V}/\delta\rho^t\rangle \mathrm{d}\boldsymbol{k}'\,\mathrm{d}\omega'$ that could appear in

(A 4) vanishes since $\langle V^t\rho^t\rangle = 0$ owing to homogeneity of the fields V^t and ρ^t.) Since the turbulence is prescribed to be isotropic and homogeneous the correlator has the form

$$\langle V^t_m(\boldsymbol{k}_2,\omega_2)\, V^t_n(\boldsymbol{k}_1,\omega_1)\rangle = Q_{mn}(\boldsymbol{k}_2,\omega_2)\,\delta(\boldsymbol{k}_1+\boldsymbol{k}_2)\,\delta(\omega_1+\omega_2), \tag{A 5}$$

$$Q_{mn}(\boldsymbol{k},\omega) = D(k,\omega)\left(\delta_{mn}-\frac{k_m k_n}{k^2}\right) - \mathrm{i}G(k,\omega)\,\epsilon_{mnl}\,k_l, \tag{A 6}$$

where functions $D(k,\omega)$ and $G(k,\omega)$ depend only on the modulus of the wave vector $\boldsymbol{k}$. When the turbulence is homogeneous, the presence of the δ-functions in (A 5) allows the Furutsu formula (A 4) to be simplified:

$$\langle \tilde{V}_j(\boldsymbol{k}_2,\omega_2)\, V^t_m(\boldsymbol{k}_1,\omega_1)\rangle = Q_{mn}(-\boldsymbol{k}_2,-\omega_2)\left\langle \frac{\delta \tilde{V}_j(\boldsymbol{k}_1,\omega_1)}{\delta V^t_n(\boldsymbol{k}_2,\omega_2)}\right\rangle. \tag{A 7}$$

In order to evaluate the functional derivative we solve (16) and (17). The Fourier transforms of (16) and (17) read

$$(-\mathrm{i}\omega+\nu_0 k^2)\,\tilde{V}_i(\boldsymbol{k},\omega) + c^2\rho_0^{-1}\,\mathrm{i}k_i\,\tilde{\rho}(\boldsymbol{k},\omega)$$
$$= \int \mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k})\,\delta(\omega_1+\omega_2-\omega)$$
$$[\mathrm{i}(k_{2m}\,\delta_{ij}+k_{1j}\,\delta_{im})\,V^t_m(\boldsymbol{k}_1,\omega_1)\,V^{(1)}_j(\boldsymbol{k}_2,\omega_2)], \tag{A 8}$$

$$-\mathrm{i}\omega\tilde{\rho}(\boldsymbol{k},\omega) + \rho_0\,\mathrm{i}k_m\,\tilde{V}_m(\boldsymbol{k},\omega)$$
$$= -\mathrm{i}k_m\int \mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k})\,\delta(\omega_1+\omega_2-\omega)$$
$$\times[\rho^{(1)}(\boldsymbol{k}_1,\omega_1)V^t_m(\boldsymbol{k}_2,\omega_2) + \rho^t(\boldsymbol{k}_2,\omega_2)V^{(1)}_m(\boldsymbol{k}_1,\omega_1)]. \tag{A 9}$$

Finding $\tilde{\rho}(\boldsymbol{k},\omega)$ from (A 9) and substituting it into (A 8) gives an equation for $\tilde{\boldsymbol{V}}(\boldsymbol{k},\omega)$ alone:

$$M_{im}\,\tilde{V}_m(\boldsymbol{k},\omega) = \mathscr{T}_i(\boldsymbol{k},\omega), \tag{A 10}$$

where the matrix M_{im} has the form

$$M_{im} = [(-\mathrm{i}\omega+\nu_0 k^2)\,\delta_{im} + \mathrm{i}c^2\omega^{-1}\,k_i\,k_m], \tag{A 11}$$

and the right-hand side of (A 10) is independent of $\tilde{\boldsymbol{V}}$:

$$\mathscr{T}_i(\boldsymbol{k},\omega) = -\mathrm{i}\int \mathrm{d}\boldsymbol{k}_1\,d\boldsymbol{k}_2\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k})\,\delta(\omega_1+\omega_2-\omega)$$
$$\times(k_{2m}\,\delta_{ij}+k_{1j}\,\delta_{im})\,V^t_m(\boldsymbol{k}_1,\omega_1)\,V^{(1)}_j(\boldsymbol{k}_2,\omega_2)$$
$$-\mathrm{i}c^2\rho_0^{-1}\,\omega^{-1}\,k_i\,k_m\int \mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k})\,\delta(\omega_1+\omega_2-\omega)$$
$$\times[\rho^{(1)}(\boldsymbol{k},\omega)\,V^t_m(\boldsymbol{k}_2,\omega_2) + \rho^t(\boldsymbol{k}_2,\omega_2)\,V^{(1)}_m(\boldsymbol{k}_1,\omega_1)]. \tag{A 12}$$

The solution of the matrix equation (A 10) is given by

$$\tilde{V}_j(\boldsymbol{k},\omega) = (-\mathrm{i}\omega+\nu_0 k^2)^{-1} M^{-1}_{ij}(\boldsymbol{k},\omega)\mathscr{T}_i(\boldsymbol{k},\omega). \tag{A 13}$$

$\boldsymbol{M}^{-1}$ in (A 13) stands for the matrix reciprocal to $\boldsymbol{M}$:

$$M^{-1}_{ij}(\boldsymbol{k},\omega) = \delta_{ij} - \beta(\boldsymbol{k},\omega)\frac{k_i k_j}{k^2}, \tag{A 14}$$

where

$$\beta(\boldsymbol{k},\omega) = \left(1 - \frac{\omega^2}{c^2 k^2} - \frac{\mathrm{i}\omega\nu_0}{c^2}\right)^{-1}. \tag{A 15}$$

Note that M_{ij}^{-1} reduces to the projection operator in the limit $c \to \infty$, $\beta \to 1$.

Taking a functional derivative of (A 13) with respect to $\boldsymbol{V}^{\mathrm{t}}$ we find the following mean value:

$$\left\langle \frac{\delta \tilde{V}_j(\boldsymbol{k},\omega)}{\delta V_n^{\mathrm{t}}(\boldsymbol{k}_2,\omega_2)} \right\rangle = (-\mathrm{i}\omega + \nu_0 k^2)^{-1} M_{ij}^{-1}(\boldsymbol{k},\omega) \left\langle \frac{\delta \mathscr{T}_i(\boldsymbol{k},\omega)}{\delta V_n^{\mathrm{t}}(\boldsymbol{k}_2,\omega_2)} \right\rangle, \tag{A 16}$$

where

$$\left\langle \frac{\delta \mathscr{T}_i(\boldsymbol{k},\omega)}{\delta V_n^{\mathrm{t}}(\boldsymbol{k}_2,\omega_2)} \right\rangle = -\mathrm{i}[(k_n - k_{2n})\,\delta_{il} + k_{2l}\,\delta_{in}]\, V_l^{(1)}(\boldsymbol{k}-\boldsymbol{k}_2, \omega-\omega_2)$$
$$- \mathrm{i}c^2 \rho_0^{-1} \omega^{-1} k_i k_n \rho^{(1)}(\boldsymbol{k}-\boldsymbol{k}_2, \omega-\omega_2). \tag{A 17}$$

With equations (A 7) and (A 16) evaluation of the average $\langle \tilde{V}_j(\boldsymbol{k}_2,\omega_2)\, V_m^{\mathrm{t}}(\boldsymbol{k}_1,\omega_1)\rangle$ is completed. Substitution in (A 2) and integration over $\boldsymbol{k}_2$ and ω_2 yields the contribution of the Reynolds stresses to the vorticity equation:

$$R_s(\boldsymbol{k},\omega) = \epsilon_{spq}\, k_p V_l^{(1)}(\boldsymbol{k},\omega)\, \sigma_{lq}(\boldsymbol{k},\omega), \tag{A 18}$$

where

$$\sigma_{lq}(\boldsymbol{k},\omega) = -\mathrm{i} \int \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\omega_1 \frac{Q_{mn}(-\boldsymbol{k}_1, -\omega_1)}{-\mathrm{i}(\omega-\omega_1) + \nu_0(\boldsymbol{k}-\boldsymbol{k}_1)^2}$$
$$\times (-k_{1j} k_{1l}\, \delta_{qm}\, \delta_{in} + k_{1j} k_n\, \delta_{qm}\, \delta_{il} - k_{1l} k_m\, \delta_{qj}\, \delta_{in} + k_m k_n\, \delta_{qj}\, \delta_{il})$$
$$\times \left[\delta_{ij} - \beta(|k-k_1|, \omega-\omega_1) \frac{(k_i - k_{1i})(k_j - k_{1j})}{(k-k_1)^2} \right]. \tag{A 19}$$

Substituting Q_{mn} in the form (A 6) in (A 20), expanding nonlinear terms in $\boldsymbol{k}_1$ and ω_1 in powers of k/k_1, dropping higher terms after integration over angular variables and performing the Fourier transformation one derives ultimately equation (18) with coefficients specified by (21) and (22).

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